NAME: ______________________ ID: _______________ DOB: __/__/19__
Department/Program: TBK / ___

Final Exam
Mechatronics (NAMO05E)
Friday, July 4, 2008 (9:00-12:00)

Please write completely your name, student ID number and date of birth on this or the first page. For all subsequent pages, you only need to write your name and the page number.

This is an open-book exam.

Question 1. (Total mark: 20)

We would like to identify components needed to build an exoskeleton system. An exoskeleton system is a device to improve the strength and endurance of human with the assistance of wearable robotic legs that support a payload. It is depicted in Figure 1 (taken from the design of Berkeley Lower exoskeleton).

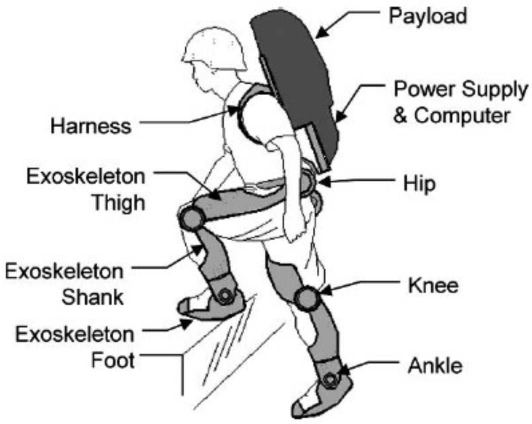


Figure 1. Exoskeleton systems (taken from the Berkeley Lower exoskeleton model).

a) Describe at least two possible objectives that must be achieved by the complete exoskeleton system. (5 marks)
b) Identify at least two possible types of actuators that can be used in the system and provide explanations for your choices (including their limitations, location of the actuators and source of auxiliary energy). (5 marks)
c) Identify all variables that should be measured (which will be used by the microprocessor to properly control the actuators) and find the at least two possible types of sensors for each measured variable. (5 marks)
d) Identify the possible reference signals that can be used by the controller (computer). (5 marks)

Answers:

a) First: Adjusting the exoskeleton in order to lift the payload.

Second: Following the movement of the human operator while maintaining the upright stability.

b) 1) Electric motor.
Limitations: Only able to provide small power when a small motor size is used.
Locations: On the exoskeleton joints.
Source of auxiliary energy: Electric source which can be fuel cell or batteries.

2) Hydraulic systems.
Limitations: It may require accummulator for maintaining sufficient pressure for standard operational. This component can be bulky.
Locations: On the exoskeleton joints.
Source of auxiliary energy: Pressure container.

3) Pneumatic systems.
Limitations: It may require accummulator for maintaining sufficient air pressure for standard operational. This component can be bulky.
Locations: On the exoskeleton joints.
Source of auxiliary energy: Air pressure container.

c) Variables to be measured:
- angular position of the actuator (exoskeleton joints)
- payload weight

Sensor to measure these variables:
- Positional sensor for the angular position. This can be optical encoder or hall sensor.
- Force or pressure sensor for measuring the payload weight. This can be strain gauge-based sensor.

d) Possible reference signal:
- angular position of the human operator joints.
- centre of gravity of the human and the exoskeleton system.

Question 2. (Total mark: 20)

In an automated factory, robotic arms is usually used for pick-and-place task. Consider the configuration in Figure 2, where a 1 degree-of-freedom robotic hand is used to sort goods from the main conveyor belt into two different conveyor belts based on their weight. There are two different goods which can be differentiated by their weight, m_1 and m_2. The robot is actuated by a shunt-wound DC motor.

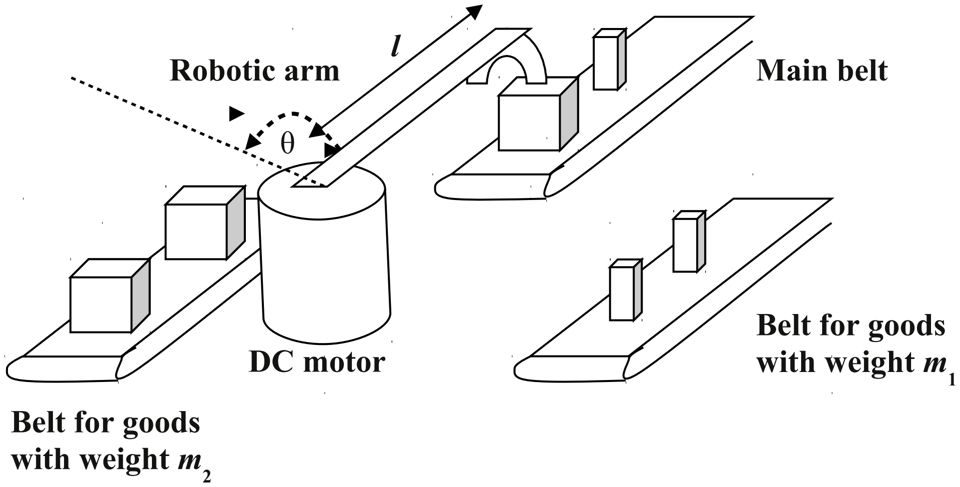


Figure 2. An automated sorting system.

a) Identify all variables that should be measured (which will be used by the microprocessor to properly control the robotic arm and to decide which belt the goods should be put) and find the at least two possible types of sensors for each measured variable. (5 marks)

b) Let the transfer function of a shunt-would DC motor from the input voltage V to the angular displacement θ be given by

$$\frac{\theta(s)}{V(s)} = \frac{\psi}{s((J+J_{load})L_A s^2 + (J+J_{load})R_A s + \psi^2)}$$

where J is the moment inertia of the motor and the robotic arm, J_{load} is the moment inertia from the load (i.e., $J_{load} = m_1 l^2$ when goods with m_1 is picked up, $J_{load} = m_2 l^2$ when goods with m_2 is collected and $J_{load} = 0$ when the arm does not carry any load). The parameters ψ, R_A and L_A are the flux linkage, the armature resistance and the armature inductance, respectively, with positive values.

Using a PI controller with transfer function $C(s) = K_p + K_i/s$, find the conditions on K_p and K_i such that the closed-loop systems remains stable for the three cases: with load m_1, with load m_2 and without any load. (*Hint: apply the Routh-Hurwitz stability test for each case to determine the interval that K_p and K_i have to satisfy.*) (15 marks)

Answers:

a) Variables to be measured:

- Weight of the load (to differentiate between no load, first load or the second load).
- Angular position of the motor (to control the direction of the arm for properly sorting the goods).

Possible type of sensors for measuring the weight: piezo electric force sensor or force spring deflection sensor.

Possible type of sensors for measuring the angular position: optical encoder, hall sensor, inductive sensing elements, etc. (see also Chapter 9 of Isermann's book).

b)

Let us denote $G(s) = \theta(s)/V(s)$, and $C(s) = K_p + K_i/s$. Now, the sensitivity transfer function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)C(s)} = \frac{(J+J_{load})L_A s^4 + (J+J_{load})R_A s^3 + \psi^2 s^2}{(J+J_{load})L_A s^4 + (J+J_{load})R_A s^3 + \psi^2 s^2 + \psi K_p s + \psi K_i}$$

So, the stability is checked by evaluating the polynomial (the denominator of the above sensitivity tranfer function):

$$\chi(s) = (J+J_{load})L_A s^4 + (J+J_{load})R_A s^3 + \psi^2 s^2 + \psi K_p s + \psi K_i$$

By using Routh-Hurwitz stability test, we need the following:

- All coefficients in χ must be positive and non-zero. This implies that $K_p>0$ and $K_i>0$.
- The first column of Routh array must not changed sign. The Routh array for χ is given by

$(J+J_{load})L_A$	ψ^2	ψK_i
$(J+J_{load})R_A$	ψK_p	0
$\dfrac{\psi^2(J+J_{load})R_A - \psi K_p(J+J_{load})L_A}{(J+J_{load})R_A}$	ψK_i	0
$\dfrac{\dfrac{\psi^2(J+J_{load})R_A - \psi K_p(J+J_{load})L_A}{(J+J_{load})R_A}\psi K_p - (J+J_{load})R_A \psi K}{\dfrac{\psi^2(J+J_{load})R_A - \psi K_p(J+J_{load})L_A}{(J+J_{load})R_A}}$	0	0
ψK_i	0	0

We now need to check the condition that makes the first column of the Routh array do not change sign.

Since the first row and the second row are positive, then in order to make the third row remains non-negative, we suffice to have

$$\frac{\psi^2(J+J_{load})R_A - \psi K_p(J+J_{load})L_A}{(J+J_{load})R_A} > 0. \qquad (1)$$

Since the denominator is always positive, then the above inequality holds if

$$\psi^2(J+J_{load})R_A - \psi K_p(J+J_{load})L_A > 0.$$

Or, equivalently,

$$K_p < \frac{\psi R_A}{L_A} \qquad (2)$$

Now, for the fourth row to remain non-negative, we only need to have the numerator to be positive (remember that the denominator is positive due to Eq. (1), that is if we choose Kp to satisfy Eq. (2)). Hence

$$\frac{\psi^2(J+J_{load})R_A - \psi K_p(J+J_{load})L_A}{(J+J_{load})R_A}\psi K_p - (J+J_{load})R_A \psi K_i > 0$$

$$\Leftrightarrow \quad K_i < \frac{\psi^2 R_A - \psi K_p L_A}{(J+J_{load})R_A^{\,2}} K_p$$

Now, the remaining condition is to have the fifth row to remain non-negative. This holds if $K_i > 0$.

Combining all the above conditions on K_p and K_i, we need the following for each cases:
Without load:

$$0 < K_p < \frac{\psi R_A}{L_A} \qquad \text{and}$$

$$0 < K_i < \frac{\psi^2 R_A - \psi K_p L_A}{J R_A^2} K_p$$

With load 1:

$$0 < K_p < \frac{\psi R_A}{L_A} \qquad \text{and}$$

$$0 < K_i < \frac{\psi^2 R_A - \psi K_p L_A}{(J + m_1 l^2) R_A^2} K_p$$

With load 2:

$$0 < K_p < \frac{\psi R_A}{L_A} \qquad \text{and}$$

$$0 < K_i < \frac{\psi^2 R_A - \psi K_p L_A}{(J + m_2 l^2) R_A^2} K_p$$

Therefore, if the PI controller needs to stabilize the system for all three cases, then we need to find the intersection of the above inequalities. This means that the common stabilizing K_p and K_i for all the three cases satisfy:

$$0 < K_p < \frac{\psi R_A}{L_A} \qquad \text{and}$$

$$0 < K_i < \min\left\{ \frac{\psi^2 R_A - \psi K_p L_A}{J R_A^2} K_p, \frac{\psi^2 R_A - \psi K_p L_A}{(J + m_1 l^2) R_A^2} K_p, \frac{\psi^2 R_A - \psi K_p L_A}{(J + m_2 l^2) R_A^2} K_p \right\}$$

Question 3. (Total mark: 20)

Consider the two-link planar manipulator as shown in Figure 3. The kinetic energy of the manipulator is given by

$$E_k = \frac{1}{2}[\dot{q}_1 \quad \dot{q}_2] \begin{bmatrix} J_{11}(q_2) & J_{21}(q_2) \\ J_{12}(q_2) & J_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

where $J_{11}(q_2) = J_1 + J_2 + m_2 l_1^2 + m_2 l_1 l_2 \cos q_2$,

$J_{12}(q_2) = J_{21}(q_2) = J_2 + \frac{m_2 l_1 l_2}{2} \cos q_2$, $J_{22} = J_2$, J_1 is the moment inertia of the first link, J_2 is the moment inertia of the second link, m_2 is the mass of the second link, l_1 and l_2 are the length of the first and second link, respectively. The potential energy is given by

$$E_p = \left(\frac{m_1 g l_1}{2} + m_2 g l_1 \right) \sin q_1 + \frac{m_2 g l_2}{2} \sin(q_1 + q_2)$$

where g is the gravitational acceleration.

a) Derive the motion of equations of the manipulator using Euler-Lagrange equations using the angle q_1 and q_2 as the generalized variables (the motion of equations should be in the form of $M(q)\ddot{q} + D(q, q)\dot{q} + G(q) = T$ where T is the torque). (20 marks)

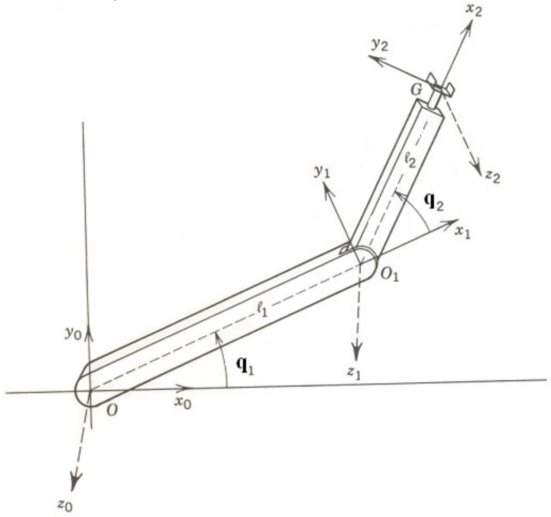


Figure 3. Two link planar manipulator.

Answers:

The Lagrangian is given by $L = E_k$-E_p. Using the kinetic and potential energy given in the question, the Lagrangian is simply:

$$L = E_k - E_p$$

$$= \frac{1}{2}\left(J_{11}(q_2)\dot{q}_1^{\,2} + 2J_{21}(q_2)\dot{q}_1\dot{q}_2 + J_{22}\dot{q}_2^{\,2} \right) - \left(\frac{m_1 g l_1}{2} + m_2 g l_1 \right) \sin q_1 - \frac{m_2 g l_2}{2} \sin(q_1 + q_2)$$

The Euler-Lagrange equation for the first generalized coordinate q_1 is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} = T_1$$

A standard computation gives us the following:

$$\frac{d}{dt}\left(J_{11}(q_2)\dot{q}_1 + J_{21}(q_2)\dot{q}_2\right) + \left(\frac{m_1 g l_1}{2} + m_2 g l_1\right)\cos q_1 + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_1$$

$$\Leftrightarrow J_{11}(q_2)\ddot{q}_1 + J_{21}(q_2)\ddot{q}_2 + \frac{\partial J_{11}(q_2)}{\partial q_2}\dot{q}_2\dot{q}_1 + \frac{\partial J_{21}(q_2)}{\partial q_2}\dot{q}_2^{\,2}$$

$$+\left(\frac{m_1 g l_1}{2} + m_2 g l_1\right)\cos q_1 + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_1$$

$$\Leftrightarrow J_{11}(q_2)\ddot{q}_1 + J_{21}(q_2)\ddot{q}_2 - m_2 l_1 l_2 \sin(q_2)\dot{q}_2\dot{q}_1 - \frac{m_2 l_1 l_2}{2}\sin(q_2)\dot{q}_2^{\,2}$$

$$+\left(\frac{m_1 g l_1}{2} + m_2 g l_1\right)\cos q_1 + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_1$$

The Euler-Lagrange equation for the second generalized coordinate q_2 is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = T_2$$

A standard computation gives us the following:

$$\frac{d}{dt}\left(J_{21}(q_2)\dot{q}_1 + J_{22}\dot{q}_2\right) - \left(\frac{1}{2}\frac{\partial J_{11}(q_2)}{\partial q_2}\dot{q}_1^{\,2} + \frac{\partial J_{21}(q_2)}{\partial q_2}\dot{q}_1\dot{q}_2\right) + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_2$$

$$\Leftrightarrow J_{21}(q_2)\ddot{q}_1 + J_{22}\ddot{q}_2 + \frac{\partial J_{21}(q_2)}{\partial q_2}\dot{q}_2\dot{q}_1 - \left(\frac{1}{2}\frac{\partial J_{11}(q_2)}{\partial q_2}\dot{q}_1^{\,2} + \frac{\partial J_{21}(q_2)}{\partial q_2}\dot{q}_1\dot{q}_2\right)$$

$$+\frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_2$$

$$\Leftrightarrow J_{21}(q_2)\ddot{q}_1 + J_{22}\ddot{q}_2 - \frac{m_2 l_1 l_2}{2}\sin(q_2)\dot{q}_1\dot{q}_2 + \frac{1}{2}m_2 l_1 l_2\sin(q_2)\dot{q}_1^{\,2} + \frac{m_2 l_1 l_2}{2}\sin(q_2)\dot{q}_1\dot{q}_2$$

$$+\frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_2$$

$$\Leftrightarrow J_{21}(q_2)\ddot{q}_1 + J_{22}\ddot{q}_2 + \frac{1}{2}m_2 l_1 l_2\sin(q_2)\dot{q}_1^{\,2} + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_2$$

So, we have the following two equations of motion:

$$J_{11}(q_2)\ddot{q}_1 + J_{21}(q_2)\ddot{q}_2 - m_2 l_1 l_2 \sin(q_2)\dot{q}_2\dot{q}_1 - \frac{m_2 l_1 l_2}{2}\sin(q_2)\dot{q}_2^{\,2}$$

$$+\left(\frac{m_1 g l_1}{2} + m_2 g l_1\right)\cos q_1 + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_1$$

and

$$J_{21}(q_2)\ddot{q}_1 + J_{22}\ddot{q}_2 + \frac{1}{2}m_2 l_1 l_2\sin(q_2)\dot{q}_1^{\,2} + \frac{m_2 g l_2}{2}\cos(q_1+q_2) = T_2$$

We combine these two equations into the form $M(q)\ddot{q} + D(q,q)\dot{q} + G(q) = T$ where

$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ as follows:

$$\begin{bmatrix} J_{11}(q_2) & J_{21}(q_2) \\ J_{21}(q_2) & J_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 \sin(q_2)\dot{q}_2 & -\frac{m_2 l_1 l_2}{2}\sin(q_2)\dot{q}_2 \\ -\frac{m_2 l_1 l_2}{2}\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$+\begin{bmatrix} \left(\frac{m_1 g l_1}{2} + m_2 g l_1\right)\cos q_1 + \frac{m_2 g l_2}{2}\cos(q_1+q_2) \\ \frac{m_2 g l_2}{2}\cos(q_1+q_2) \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

Question 4. (Total mark: 40)

A Segway transportation vehicle, shown in Figure 4(a), is a two-wheeled device that can be steered by controlling the angle of the steering bar. When it moves forward or backward, it can be modeled as an inverted pendulum as depicted in Figure 4(b). The pivot of the pendulum is mounted on a base that can move in a horizontal direction. The base is mounted on a wheel driven by a motor that exerts torque T. The mass of the base and wheel is M and the mass of the pendulum is m. Assume that the wheel is not slipping on the surface, i.e., the horizontal force $F = T/R$ is exerted on the base where R is the radius of the wheel.

In order to use the Newtonian approach, the inverted pendulum can be decomposed into two components as illustrated in Figure 4(c) and 4(d).

In Figure 4(c), there are two forces acting on the base, F being the force exerted by the motor and H being the reaction force from the pendulum.

Figure 4(d) shows the forces acting on the pendulum, which are mg at the center of mass, the reaction horizontal force from the base H, the vertical reaction force from the base V at the pivot of the pendulum, the horizontal force $T/L \cos\theta$ and vertical force $T/L \sin\theta$ at the center of mass due to the reaction torque from the wheel. The distance from the center of mass to the pivot is given by L.

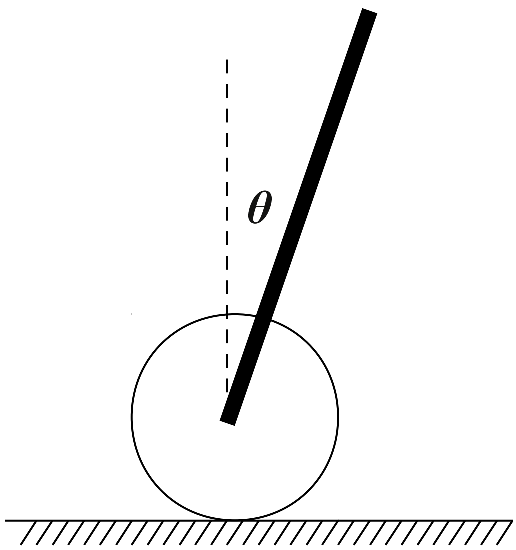
a) Using the diagram in Figure 4(d), write down the horizontal and vertical Newton's laws of the pendulum at the center of mass. (Note that the coordinate of centre of mass is $(x+L\sin\theta, L\cos\theta)$.) (Marks: 4).
b) Using the diagram in Figure 4(d), write down the rotational motion of the pendulum around the center of mass. (You can use J as the moment inertia of the pendulum). (Marks: 3).
c) Using the diagram in Figure 4(c), write down the horizontal Newton's laws of the base. (Marks: 3).
d) Based on these three equations in the above questions, shows that the equations of motion reduce to

$$\begin{bmatrix} J+mL^2 & mL\cos\theta \\ mL\cos\theta & m+M \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ -mL\dot{\theta}\sin\theta & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix}+\begin{bmatrix} -mgL\sin\theta \\ 0 \end{bmatrix}=\begin{bmatrix} -T \\ \frac{T}{R}-\frac{T}{L}\cos\theta \end{bmatrix}$$

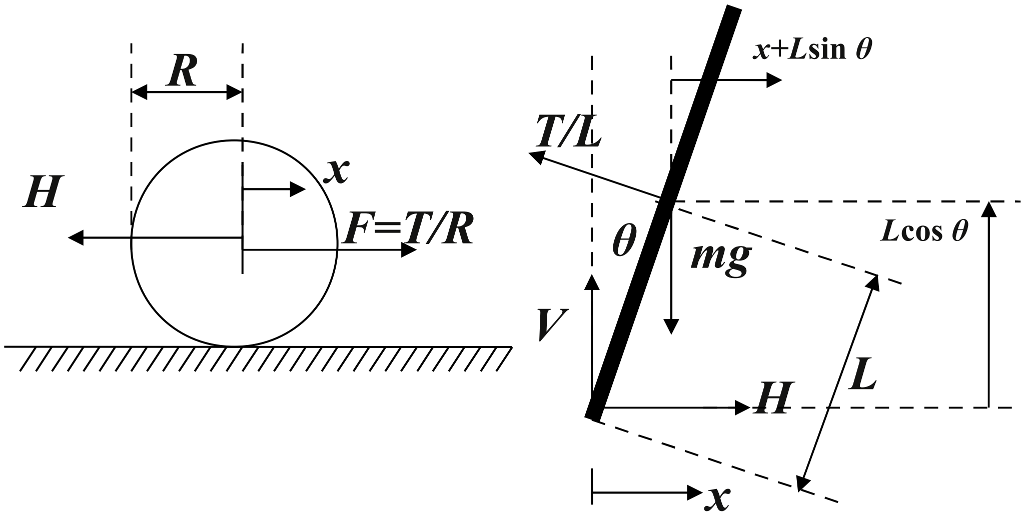
(Marks: 20).

e) Write down the state equations with T as the input and θ as the measurement output. (Marks: 10).

(a)



(b)



(c) (d)

Figure 4. Modeling the Segway transportation system.

Answers:

a)

The horizontal Newton's law is given by

$$H - \frac{T}{L}\cos\theta = m\frac{d^2(x + L\sin\theta)}{dt^2}$$

$$\Leftrightarrow H - \frac{T}{L}\cos\theta = m\ddot{x} + m\frac{d}{dt}(L\cos(\theta)\dot{\theta})$$

$$\Leftrightarrow H - \frac{T}{L}\cos\theta = m\ddot{x} + mL\cos(\theta)\ddot{\theta} - mL\sin(\theta)\dot{\theta}^2$$

Similarly, the vertical Newton's law is given by

$$V + \frac{T}{L}\sin\theta - mg = m\frac{d^2(L\cos\theta)}{dt^2}$$

$$\Leftrightarrow V + \frac{T}{L}\sin\theta - mg = m\frac{d}{dt}(-L\sin(\theta)\dot{\theta})$$

$$\Leftrightarrow V + \frac{T}{L}\sin\theta - mg = -mL\sin(\theta)\ddot{\theta} - mL\cos(\theta)\dot{\theta}^2$$

b)

Rotational motion around the centre of mass is given by

$$J\ddot{\theta} = VL\sin\theta - HL\cos\theta$$

c)

The horizontal motion of the base is given by

$$M\ddot{x} = \frac{T}{R} - H$$

d)

The four equations that we have, so far, are

$$H - \frac{T}{L}\cos\theta = m\ddot{x} + mL\cos(\theta)\ddot{\theta} - mL\sin(\theta)\dot{\theta}^2 \tag{3}$$

$$V + \frac{T}{L}\sin\theta - mg = -mL\sin(\theta)\ddot{\theta} - mL\cos(\theta)\dot{\theta}^2 \tag{4}$$

$$J\ddot{\theta} = VL\sin\theta - HL\cos\theta \tag{5}$$

$$M\ddot{x} = \frac{T}{R} - H \tag{6}$$

Substituting (3) to (6) gives us

$$M\ddot{x} = \frac{T}{R} - \frac{T}{L}\cos\theta - m\ddot{x} - mL\cos(\theta)\ddot{\theta} + mL\sin(\theta)\dot{\theta}^2$$

$$\Leftrightarrow (M+m)\ddot{x} + mL\cos(\theta)\ddot{\theta} - mL\sin(\theta)\dot{\theta}^2 = \frac{T}{R} - \frac{T}{L}\cos\theta \tag{7}$$

Substituting (3) and (4) into (5), gives us

$$J\ddot{\theta} = \left(-\frac{T}{L}\sin\theta + mg - mL\sin(\theta)\ddot{\theta} - mL\cos(\theta)\dot{\theta}^2\right)L\sin\theta$$
$$-\left(\frac{T}{L}\cos\theta + m\ddot{x} + mL\cos(\theta)\ddot{\theta} - mL\sin(\theta)\dot{\theta}^2\right)L\cos\theta$$
$$\Leftrightarrow J\ddot{\theta} = -T\sin^2(\theta) + mgL\sin(\theta) - mL^2\sin^2(\theta)\ddot{\theta} - mL^2\sin(\theta)\cos(\theta)\dot{\theta}^2$$
$$-T\cos^2(\theta) - mL\cos\theta\ddot{x} - mL^2\cos^2(\theta)\ddot{\theta} + mL^2\sin(\theta)\cos(\theta)\dot{\theta}^2$$
$$\Leftrightarrow J\ddot{\theta} = -T + mgL\sin(\theta) - mL^2\ddot{\theta} - mL\cos\theta\ddot{x}$$
$$\Leftrightarrow \left(J + mL^2\right)\ddot{\theta} + mL\cos\theta\ddot{x} - mgL\sin(\theta) = -T \tag{8}$$

where the second equivalence (equation) is due to the fact that $\sin^2\theta + \cos^2\theta = 1$.

The combination of the above two equations of motion from (7) and (8) can be written also in the following form:

$$\begin{bmatrix} J + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -mL\dot{\theta}\sin\theta & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} -mgL\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -T \\ \frac{T}{R} - \frac{T}{L}\cos\theta \end{bmatrix}$$

e)

Note that the equations of motion found in the question part d) can also be written as:

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{x}\end{bmatrix}=\begin{bmatrix}J+mL^2 & mL\cos\theta\\ mL\cos\theta & m+M\end{bmatrix}^{-1}\left(-\begin{bmatrix}0 & 0\\ -mL\dot{\theta}\sin\theta & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{x}\end{bmatrix}-\begin{bmatrix}-mgL\sin\theta\\ 0\end{bmatrix}+\begin{bmatrix}-T\\ \frac{T}{R}-\frac{T}{L}\cos\theta\end{bmatrix}\right)$$

$$=\frac{1}{\Delta(\theta)}\begin{bmatrix}m+M & -mL\cos\theta\\ -mL\cos\theta & J+mL^2\end{bmatrix}\left(-\begin{bmatrix}0 & 0\\ -mL\dot{\theta}\sin\theta & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{x}\end{bmatrix}-\begin{bmatrix}-mgL\sin\theta\\ 0\end{bmatrix}+\begin{bmatrix}-T\\ \frac{T}{R}-\frac{T}{L}\cos\theta\end{bmatrix}\right)$$

$$=\frac{1}{\Delta(\theta)}\begin{bmatrix}(m+M)(mgL\sin\theta-T)-mL\cos(\theta)\left(\frac{T}{R}-\frac{T}{L}\cos\theta+mL\dot{\theta}^2\sin\theta\right)\\ (-mL\cos(\theta))(mgL\sin\theta-T)+\left(J+mL^2\right)\left(\frac{T}{R}-\frac{T}{L}\cos\theta+mL\dot{\theta}^2\sin\theta\right)\end{bmatrix}$$

$$=\frac{1}{\Delta(\theta)}\begin{bmatrix}(m+M)(mgL\sin\theta)-mL\cos(\theta)\left(mL\dot{\theta}^2\sin\theta\right)\\ (-mL\cos(\theta))(mgL\sin\theta)+\left(J+mL^2\right)\left(mL\dot{\theta}^2\sin\theta\right)\end{bmatrix}+\frac{1}{\Delta(\theta)}\begin{bmatrix}-(m+M)-mL\cos(\theta)\left(\frac{1}{R}-\frac{1}{L}\cos\theta\right)\\ mL\cos(\theta)+\left(J+mL^2\right)\left(\frac{1}{R}-\frac{1}{L}\cos\theta\right)\end{bmatrix}$$

where $\Delta(\theta)=(J+mL)(m+M)-(mL\cos\theta)^2$ is the determinant of $\begin{bmatrix}J+mL^2 & mL\cos\theta\\ mL\cos\theta & m+M\end{bmatrix}$.

Now, using $x_1=\theta, x_2=\dot{\theta}, x_3=x$ and $x_4=\dot{x}$, the state equation can be written as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}x_2\\ \frac{(m+M)(mgL\sin x_1)-mL\cos(x_1)\left(mLx_2{}^2\sin x_1\right)}{\Delta(x_1)}+\frac{-(m+M)-mL\cos(x_1)\left(\frac{1}{R}-\frac{1}{L}\cos x_1\right)}{\Delta(x_1)}T\\ x_4\\ \frac{(-mL\cos(x_1))(mgL\sin x_1)+\left(J+mL^2\right)\left(mLx_2{}^2\sin x_1\right)}{\Delta(x_1)}+\frac{mL\cos(x_1)+\left(J+mL^2\right)\left(\frac{1}{R}-\frac{1}{L}\cos x_1\right)}{\Delta(x_1)}T\end{bmatrix}$$